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An Application of DHW Algorithm for the Solution of Inverse Heat Conduction Problem[1]

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A damped heat wave (DHW) algorithm is applied for the temperature distribution calculation in a solution of a linear inverse heat conduction problem (IHCP). A nonlinear least squares algorithm is used for calculation of the unknown boundary heat flux history in a one-dimensional medium. The solution is based on the assumption that the temperature measurements are available, at least, at one point of the medium over the whole time domain. Sample calculations, for a comparison between exact heat sources and estimated ones, are made to confirm the validity of the proposed method. The close agreement between the exact and estimated values calculated for both exact and noisy data shows the potential of the proposed method for finding a relatively accurate heat source distribution in a one-dimensional homogeneous finite medium. The proposed method of solving inverse heat conduction problems is very simple and easy to implement.

KEY WORDS: inverse heat conduction problem; numerical algorithm; temperature distribution.

1. INTRODUCTION

One of the important types of inverse heat conduction problems (IHCP) deals with the determination of the boundary heat flux history from the known transient temperature distribution in a solid.

Existing methods for the calculation of the heat flux from the temperature rise of the sample are based on an analytical or numerical solution

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of the heat conduction equation:

$$\rho c \frac{\partial T}{\partial t} = \nabla(\lambda \nabla T), \tag{1}$$

where ρ is the density, c is the specific heat, λ is the thermal conductivity, ∇ is the Hamilton operator, and $T = T(\vec{r}, t)$ is the temperature in a space-time point $(\vec{r}, t)$. Equation (1) is solved with certain initial and boundary conditions, and the solution of this direct heat conduction problem (DHCP) is then compared with the experimental temperature rise. Although an IHCP is an ill-posed problem, the heat flux function can usually be estimated from the comparison. Fundamental concepts of inverse heat conduction, and an extensive bibliography and survey on IHCP methods can be found in Beck et al. [1] and in Alifanov [2].

The present work addresses the unsteady linear IHCP in a finite medium with a time-variable heat flux at the boundary. A numerical solution of the DHCP is obtained using our damped heat wave (DHW) algorithm described in Refs. 3 and 4. The test input data for IHCP were generated using an explicit finite difference (EFD) algorithm. The temperature points at discrete regular times were modified by adding random errors produced by a random number generator. Three examples of the temperature history for piece-wise heat flux functions are given in this paper to illustrate the use of the algorithm even in cases of very noisy temperature history signals.

The DHW algorithm is very simple, universal, and easy to implement in comparison with most analytical or numerical solutions of Eq. (1).

2. DHW ALGORITHM DESCRIPTION

In the DHW algorithm for the calculation of the temperature distribution in one dimension, the medium [3] is divided into N equal slabs of thickness $\Delta l = L/N$. These slabs are replaced by a perfect conductor of the same heat capacity separated by a thermal resistance $\Delta l/\lambda$ (where λ is the thermal conductivity of the medium), so the temperature within a slab at any given time is constant. Heat propagates from one slab to another due to the existence of a temperature difference between the slabs. A certain portion (given by the inner transfer coefficient ξ) of the excessive heat energy moves from one slab to the next one, lowering thus the temperature difference between the two neighbor slabs. This redistribution process (called the damped heat wave) starts from the left boundary slab and marches in space from one pair of slabs to another. When the wave reaches the boundary of the medium, it bounces back and moves in the opposite direction in a perpetual manner.

The inner transfer coefficient ξ is a dimensionless quantity given by [4]

$$\xi = \frac{Fo^*}{Fo^* + 2}, \tag{2}$$

where

$$Fo^* = \frac{\alpha \Delta t}{\Delta l^2} \tag{3}$$

is the Fourier number for one slab and α is the thermal diffusivity of the slab material. The time step Δt is equal to one loop time interval of the heat wave.

When the wave imitates diffusion, the upper limit for the inner transfer coefficient is $\xi < 0.5$. It follows from Eqs. (2) and (3) that the upper limit for the time step Δt is then given by

$$\Delta t < \frac{2\Delta l^2}{\alpha}. \tag{4}$$

This introduces a limit to the maximum size of the time step that can be chosen for a fixed Δl. The wave speed $v = 2L/\Delta t$ can be chosen arbitrarily, but from Eq. (4) we get $v > \alpha N^2/L$. Generally, the calculated distribution using waves with a higher speed is more precise than that with the lower ones.

In case of heat losses a part of the excessive thermal energy leaves the medium each time the wave reaches the boundary slabs. The temperature change ΔT of the boundary slab due to heat losses is

$$\Delta T = -\zeta (T - T_{\mathrm{a}}), \tag{5}$$

where ζ is the surface transfer coefficient and T_{a} is the ambient temperature. The surface transfer coefficient is a dimensionless quantity defined as

$$\zeta = Bi\, Fo^*, \tag{6}$$

where Bi is the Biot number for one slab ($Bi = hL/\lambda N$ where h is the coefficient of surface heat transfer). Generally, there are two different surface transfer coefficients for a finite medium, one for each surface. In order to fulfill the limitation $\zeta \in \langle 0, 1\rangle$, the time step Δt has to be limited to

$$\Delta t \leqslant \frac{\Delta l^2}{\alpha Bi}. \tag{7}$$

For $Bi < 0.5$ the condition given in Eq. (4) is more restrictive than in Eq. (7), and is actually limiting the time step value.

3. INVERSE PROBLEM FORMULATION

Consider a medium of thickness L and constant thermal properties, originally at zero temperature. At a specific time, $t = 0$, a heat flux is applied to one surface at $x = 0$. The temperature history is measured on the opposite surface at $x = L$. We will use the dimensionless time (the Fourier number) defined as $Fo = \alpha t/L^2$ instead of the time t. The heat flux function q is calculated in certain discrete dimensionless time points $Fo_j = \alpha t_j/L^2$, $j = 0, 1, 2, 3, \ldots, n$.

An ordinary least squares procedure [5] was used to find the unknown parameters q_j from

$$\min_{q_j} \sum_{i=1}^{k} \left[T_i(Fo_i, q_j) - Y_i\right]^2, \tag{8}$$

where $T_i(Fo_i, q_j)$ is the temperature point at time Fo_i calculated using the DHW algorithm and Y_i, $i = 1, 2, 3, \ldots, k$ are the points of the temperature response curve (observed data). More than one set of temperature history points can be used to find the heat flux components in this procedure. It means that the temperature response can be measured at the same time in different locations of the medium. Different weights can be also assigned to different temperature points in order to improve the precision of the calculation. A weighted orthogonal distance regression algorithm [5] can be used for heat flux estimation in the case when errors are expected in both temperature and time data.

4. INVERSE PROBLEM SOLUTION

4.1. Sensitivity Analysis

In order to find conditions for an optimal experiment design, we have to analyze sensitivity coefficients of the IHCP. Dimensionless sensitivity coefficients $X_j(Fo)$ are defined [1, p. 30] as the first partial derivative of temperature $T = T(Fo, q_j)$ with respect to the heat flux component q_j

$$X_j(Fo) \equiv \frac{\lambda}{L} \frac{\partial T}{\partial q_j}, \quad j = 0, 1, 2, 3, \ldots, n. \tag{9}$$

The heat flux values between the discrete time points $q_j = q(Fo_j)$, where $Fo_j = j\Delta Fo$, $j = 0, 1, 2, 3, \ldots, n$, are linearly interpolated. The component q_0 is

$$q_0(Fo) = \begin{cases} 0, & Fo < 0 \\ q_0 + (q_1 - q_0)\frac{Fo}{Fo_1}, & Fo \leqslant Fo_1 \\ 0, & Fo > Fo_1 \end{cases} \tag{10}$$

For q_j, $j = 1, 2, 3, \ldots, n-1$ we have

$$q_j(Fo) = \begin{cases} 0, & Fo < Fo_{j-1} \\ q_{j-1} + (q_j - q_{j-1})\frac{Fo - Fo_{j-1}}{Fo_j - Fo_{j-1}}, & Fo_{j-1} \leqslant Fo \leqslant Fo_j \\ q_j + (q_{j+1} - q_j)\frac{Fo - Fo_j}{Fo_{j+1} - Fo_j}, & Fo_j \leqslant Fo \leqslant Fo_{j+1} \\ 0, & Fo > Fo_{j+1} \end{cases} \tag{11}$$

In order to identify an abruptly changing heat flux on the medium surface, the number of heat flux points n should be as high as possible, with ΔFo small. On the other hand, due to the diffusion nature of heat propagation, this requirement is in contradiction with the identifiability of the heat flux components. As $\Delta Fo \to 0$, the sensitivity coefficients $X_j \to 0$, and, furthermore, they become linearly dependent. The problem of simultaneous heat flux component identification from temperature history is difficult in this case and very sensitive to measurement errors. Usually, an optimal value (or range) of ΔFo can be found, for which enough heat flux components can be estimated with a reasonable precision to identify an unknown piece-wise heat flux function.

The heat flux sensitivity coefficients for a surface temperature in a homogeneous finite medium for $0 < Fo < 2.5$, $\Delta Fo = 0.5$, and $Bi = 0.1$ are plotted in Fig. 1. The first coefficient X_0 is only half of the amplitude of the coefficients with higher j, and its shape also differs from the others. Fortunately, the precision of the first heat flux component determination can be enhanced using additional information about the initial conditions of the problem at $Fo = 0$. The last two coefficients are affected by the fact that it takes about $2\Delta Fo$ for the sensitivity coefficient to reach its maximum value. It is therefore better to measure the temperature long enough, to be able to make an additional assumption about the value of the last heat flux component.

The first six sensitivity coefficients for the medium with $0 < Fo < 2.5$, $\Delta Fo = 0.25$, and $Bi = 0.1$ are plotted in Fig. 2. The coefficients are closer to each other and rising steeper than those in Fig. 1. Their magnitudes are

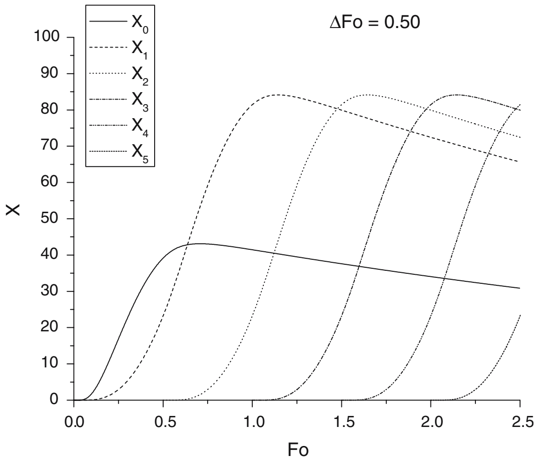

Fig. 1. Sensitivity coefficients X_i for $\Delta Fo = 0.5$.

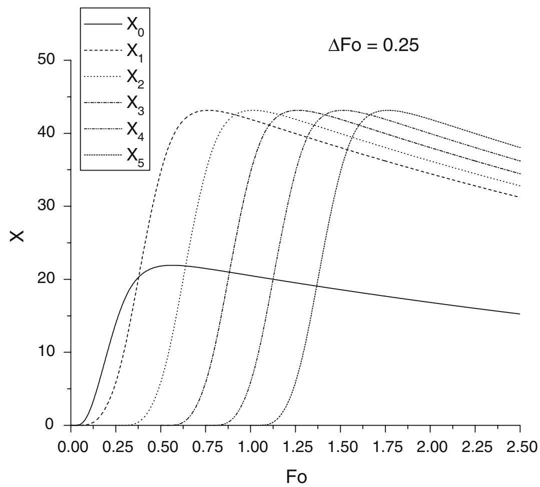

Fig. 2. Sensitivity coefficients X_i for $\Delta Fo = 0.25$.

directly proportional to ΔFo. The identifiability of the coefficients, especially for a noisy signal, is much smaller than in the case of $\Delta Fo = 0.5$.

For stable heat flux component calculations, $\Delta Fo \geqslant 0.4$ should be chosen in the total time domain $Fo \geqslant 4$. In this case, the first ten heat flux components can be calculated with good accuracy.

It has to be noted that the optimal value of ΔFo depends also on the heat pulse shape and its duration. Sharp and abrupt changes of the heat

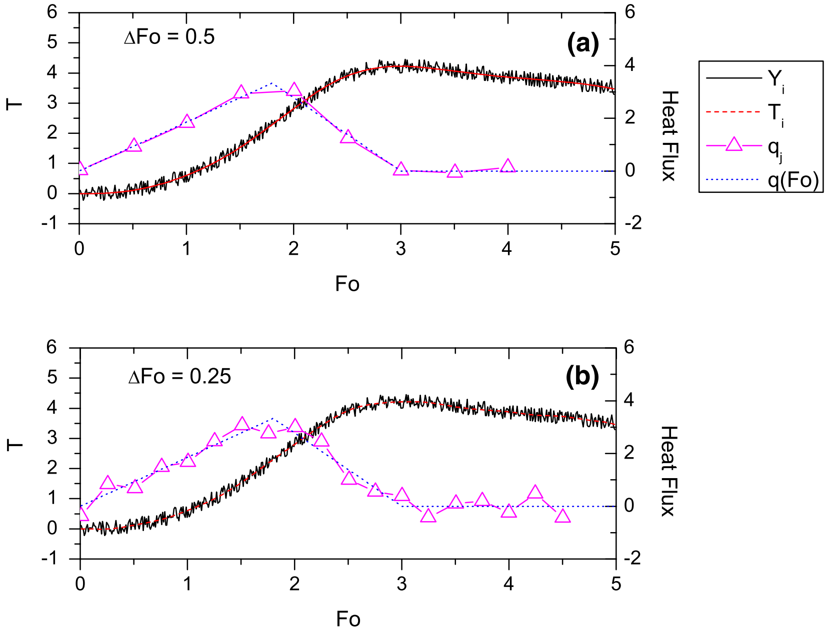

Fig. 3. Temperature history and calculated heat flux components for a triangular shape heat pulse: (a) $\Delta Fo = 0.5$, and (b) $\Delta Fo = 0.25$.

flux can lead to excessive oscillations in the calculated components, which is characteristic for all ill-posed problems.

The calculated heat flux components accuracy does not depend on heat losses. The described IHCP procedure can be applied to the temperature signal without prior knowledge of the heat loss coefficient. It is assumed, that the medium thickness L and the thermal diffusivity α are known parameters.

4.2. Examples

4.2.1. Example 1

An example of the triangular heat pulse is given in Fig. 3. The medium was divided into 20 parts. A total of 525 temperature vs. time points were generated using the EFD algorithm with a heat pulse which started at $Fo = 0$, with a peak at $Fo = 1.8$, and ended at $Fo = 3$. The Biot number was $Bi = 0.1$ and the noise level was set to 0.2 of the units on the temperature scale. The temperature history data and the results of the heat

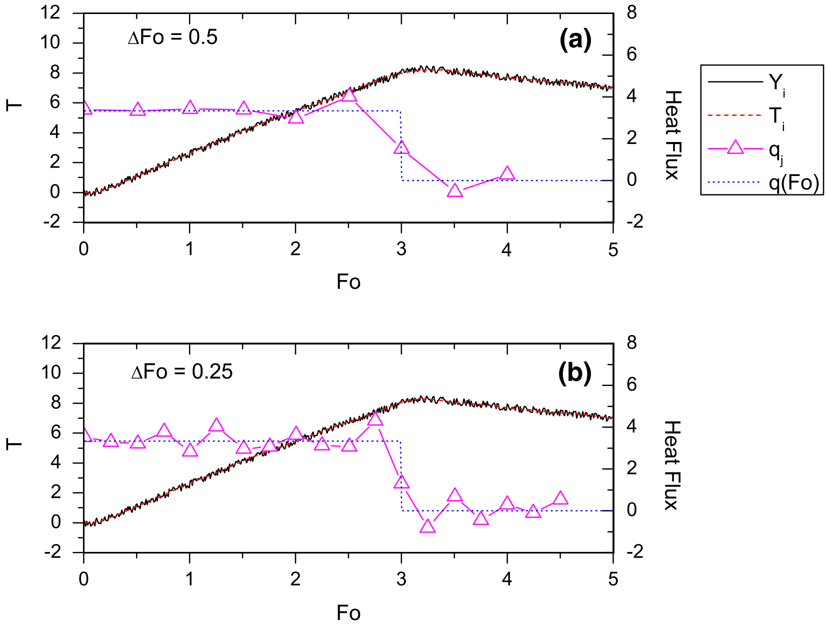

Fig. 4. Temperature history and calculated heat flux components for a rectangular shape heat pulse: (a) $\Delta Fo = 0.5$, and (b) $\Delta Fo = 0.25$.

flux component calculation for $\Delta Fo = 0.5$ and $\Delta Fo = 0.25$ are plotted in Figs. 3a and 3b, respectively.

The agreement between theoretical and calculated values of the heat flux components is very good, especially for the case of $\Delta Fo = 0.5$. The last two heat flux components from the end of the time interval were intentionally omitted, due to the reasons mentioned above.

4.2.2. Example 2

The second example data were generated in the same manner as in Example 1, but for a rectangular heat pulse, which started at $Fo = 0$ and ended at $Fo = 3$. The noise level was set to 0.5. Data and results of the heat flux component calculation are plotted in Fig. 4a,b. An abrupt change in the heat flux at $Fo = 3$ caused significant oscillations in the calculated heat flux components, both for $\Delta Fo = 0.5$ and $\Delta Fo = 0.25$. The first step change of the heat flux at $Fo = 0$ was effectively damped by the fact that the initial condition $T = 0$, $Fo = 0$ can be used as additional information in the temperature calculations.

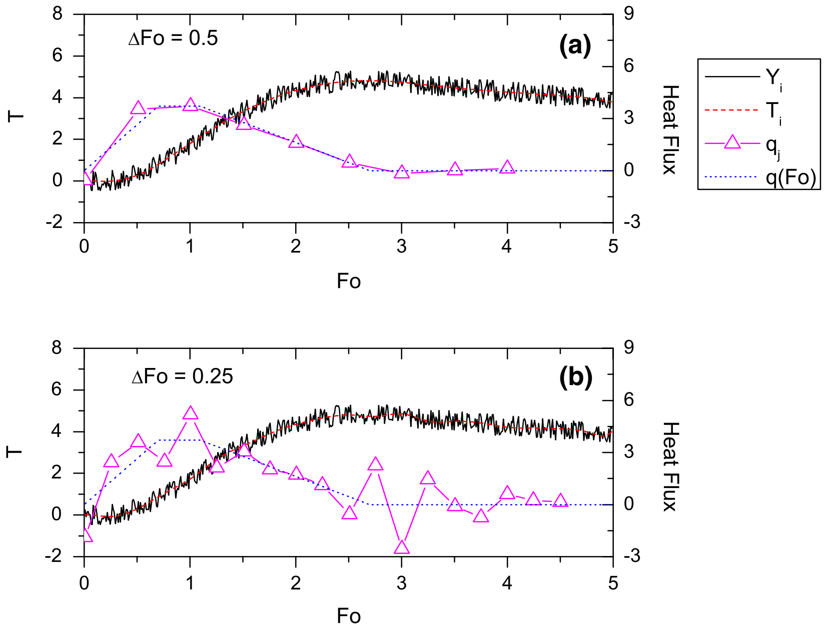

Fig. 5. Temperature history and calculated heat flux components for a trapezoid shape heat pulse: (a) $\Delta Fo = 0.5$, and (b) $\Delta Fo = 0.25$.

4.2.3. Example 3

The temperature history for a trapezoidal heat pulse, which started at $Fo = 0$, duration to $Fo = 2.7$, was generated in the third example. The noise level was set to 1. The temperature and the results of the heat flux component calculations are plotted in Fig. 5. Even for such a noisy signal, the results for heat flux components for $\Delta Fo = 0.5$ are in very good agreement with the theoretically expected values (see Fig. 5a). On the contrary, the results for heat flux components for $\Delta Fo = 0.25$, plotted in Fig. 5b, are obviously too scattered for a realistic reconstruction of the heat pulse.

5. CONCLUSION

A one-dimensional IHCP in a finite medium can be solved using a simple iterative DHW algorithm. In this algorithm, the temperature is calculated explicitly in one simple calculation that is repeated for each time step as the heat wave marches through the medium with a constant speed.

The proposed algorithm is quite stable and the heat flux function can be reconstructed even in the case of relatively noisy temperature signals.

The algorithm can be used as a fast, easy-to-understand and easy-to-implement alternative to existing analytical and numerical methods to solve inverse heat conduction problems in a finite, one-dimensional, homogeneous medium.

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